MATH 2230 HW4

P. 119

2(a) $\int_0^1 (1+it)^2\,dt = \int_0^1 1-t^2\,dt + i\int_0^1 2t\,dt = [t-\frac{1}{3}t^3]_0^1 + i[t^2]_0^1 = \frac{2}{3}+i$

(b) $\int_1^2 (\frac{1}{t}-i)^2\,dt = \int_1^2 \frac{1}{t^2}-1\,dt - i\int_1^2 \frac{2}{t}\,dt = [-\frac{1}{t}-t]_1^2 - i[2\ln t]_1^2 = -\frac{1}{2} - i\ln 4$

(c) $\int_0^{\frac{\pi}{6}} e^{i2t}\,dt = [\frac{1}{2i}e^{i2t}]_0^{\frac{\pi}{6}} = \frac{1}{2i}[\frac{1}{2}+\frac{\sqrt{3}}{2}i-1] = \frac{\sqrt{3}}{4}+\frac{i}{4}$

(d) $\int_0^\infty e^{-zt}\,dt = \lim_{h\to\infty}\int_0^h e^{-zt}\,dt = \lim_{h\to\infty}[\frac{e^{-zt}}{-z}]_0^h = \lim_{h\to\infty}\frac{e^{-zh}}{-z}+\frac{1}{z} = \frac{1}{z}$

3. If $m=n$, $\int_0^{2\pi} e^{im\theta}\cdot e^{-in\theta}\,d\theta = \int_0^{2\pi} d\theta = 2\pi$

If $m\neq n$, $\int_0^{2\pi} e^{im\theta}e^{-in\theta}\,d\theta = \int_0^{2\pi} e^{i(m-n)\theta}\,d\theta = [\frac{1}{i(m-n)}e^{i(m-n)\theta}]_0^{2\pi} = \frac{1}{i(m-n)}(1-1) = 0.$

P. 124

5. Let $f(z) = u(x,y)+iv(x,y)$, $z(t) = x(t)+iy(t)$

$$\begin{aligned} w'(t) &= \frac{d}{dt}u(x(t),y(t)) + i\frac{d}{dt}v(x(t),y(t)) \\ &= u_x x'(t) + u_y y'(t) + i(v_x x'(t) + v_y y'(t)) \\ &= u_x x'(t) - v_x y'(t) + i(v_x x'(t) + u_x y'(t)) \\ &= u_x(x'(t)+iy'(t)) + iv_x(x'(t)+iy'(t)) \\ &= (u_x+iv_x)(x'(t)+iy'(t)) \\ &= f'(z(t))\,z'(t) \end{aligned}$$

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1(a) $z'(\theta) = 2ie^{i\theta}$

$\int_C f(z)\,dz = \int_0^\pi \frac{2e^{i\theta}+2}{2e^{i\theta}}\cdot 2ie^{i\theta}\,d\theta = 2\int_0^\pi ie^{i\theta}+i\,d\theta = 2[e^{i\theta}+i\theta]_0^\pi = -4+2\pi i$

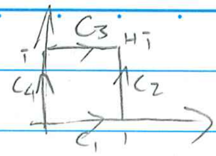
(b) $z'(\theta) = 2ie^{i\theta}$

$\int_C f(z)\,dz = \int_\pi^{2\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}}\cdot 2ie^{i\theta}\,d\theta = 2[e^{i\theta}+i\theta]_\pi^{2\pi} = 4+2\pi i$

(c) Assume the two curves in (a) and (b) are C_1 and C_2 respectively.

$\int_C f(z)\,dz = (\int_{C_1}+\int_{C_2}) f(z)\,dz = -4+2\pi i+4+2\pi i = 4\pi i$

3. On C_1, $z(t) = t$, $0 \le t \le 1$



$$\int_{C_1} f(z)\,dz = \int_0^1 \pi \exp(\pi t)\cdot 1\,dt = e^{\pi t}\Big|_0^1 = e^{\pi} - 1$$

On C_2, $z(t) = 1 + it$, $0 \le t \le 1$

$$\int_{C_2} f(z)\,dz = \int_0^1 \pi \exp(\pi(1-it))(i)\,dt = e^{\pi}\int_0^1 \pi i e^{-i\pi t}\,dt = e^{\pi}\left[-e^{-i\pi t}\right]_0^1 = 2e^{\pi}$$

On C_3, $z(t) = t + i$, $0 \le t \le 1$

$$\int_{C_3} f(z)\,dz = \int_0^1 \pi \exp((t-i)\pi)\cdot 1\,dt = -\int_0^1 \pi e^{\pi t}\,dt = -e^{\pi t}\Big|_0^1 = 1 - e^{\pi}$$

On C_4, $z(t) = it$, $0 \le t \le 1$

$$\int_{C_4} f(z)\,dz = \int_0^1 \pi \exp(\pi(-it))\cdot i\,dt = -e^{-i\pi t}\Big|_0^1 = 2$$

$$\int_C f\,dz = \left(\int_{C_1} + \int_{C_2} - \int_{C_3} - \int_{C_4}\right) f\,dz$$
$$= e^{\pi} - 1 + 2e^{\pi} - (1 - e^{\pi}) - 2$$
$$= 4(e^{\pi} - 1)$$

7. $$\int_C f(z)\,dz = \int_0^{\frac{\pi}{2}} \exp\left[(-1-2i)\,\mathrm{Log}\,e^{i\theta}\right] ie^{i\theta}\,d\theta$$
$$= \int_0^{\frac{\pi}{2}} \exp\left[(-1-2i)i\theta\right] ie^{i\theta}\,d\theta$$
$$= \int_0^{\frac{\pi}{2}} \exp\left[2\theta - i\theta\right] ie^{i\theta}\,d\theta$$
$$= i\int_0^{\frac{\pi}{2}} e^{2\theta}\,d\theta$$
$$= i\,\frac{e^{\pi} - 1}{2}$$

9(a) On C, let $z(\theta) = e^{i\theta}$, $-\pi < \theta \le \pi$

$$\int_C f(z)\,dz = \int_{-\pi}^{\pi} \exp\left[-\tfrac{3}{4}\,\mathrm{Log}\,e^{i\theta}\right] ie^{i\theta}\,d\theta$$
$$= \int_{-\pi}^{\pi} \exp\left[-\tfrac{3}{4} i\theta\right] ie^{i\theta}\,d\theta$$
$$= \int_{-\pi}^{\pi} i\exp\left(\tfrac{1}{4} i\theta\right)\,d\theta$$
$$= 4\exp\left(\tfrac{1}{4} i\theta\right)\Big|_{-\pi}^{\pi}$$
$$= 4\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i - \left(\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i\right)\right) = 4\sqrt{2}\,i$$

(b) On C, let $z(\theta) = e^{i\theta}$, $0 < \theta \le 2\pi$

$$\int_C f(z)\,dz = \int_0^{2\pi} \exp\left[-\tfrac{3}{4}\log e^{i\theta}\right] ie^{i\theta}\,d\theta$$
$$= \int_0^{2\pi} \exp\left[-\tfrac{3}{4} i\theta\right] ie^{i\theta}\,d\theta$$
$$= \int_0^{2\pi} i\exp\left[i\tfrac{\theta}{4}\right]\,d\theta$$
$$= 4\exp\left(i\tfrac{\theta}{4}\right)\Big|_0^{2\pi}$$
$$= 4(i - 1) = -4 + 4i$$

10. Let $z(\theta) = e^{i\theta}$, $0 < \theta \le 2\pi$

$$\int_C z^m \bar{z}^n \, dz = \int_0^{2\pi} e^{im\theta} e^{-in\theta} \cdot ie^{i\theta} \, d\theta$$
$$= i \int_0^{2\pi} e^{i(m+1)\theta} e^{-in\theta}$$
$$= \begin{cases} 0 & \text{if } m+1 \neq n \\ 2\pi i & \text{if } m+1 = n \end{cases}$$

13. $$\int_{C_0} (z - z_0)^{n-1} \, dz = \int_{-\pi}^{\pi} (z_0 + Re^{i\theta} - z_0)^{n-1} \cdot iRe^{i\theta} \, d\theta$$
$$= \int_{-\pi}^{\pi} R^n e^{in\theta} \, d\theta$$
$$= \begin{cases} 0 & \text{when } n = \pm 1, \pm 2 \\ 2\pi i & \text{when } n = 0 \end{cases}$$ by Q3 of p.119

P.138-139

1(a) $$\left| \frac{z+4}{z^3-1} \right| \le \frac{|z|+4}{|z|^3-1} = \frac{2+4}{2^3-1} = \frac{6}{7}$$ on C

Length of $C = \pi$

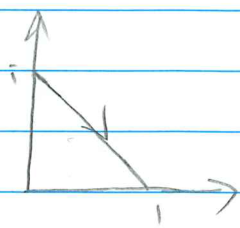
$$\therefore \left| \int_C \frac{z+4}{z^3-1} \, dz \right| \le \frac{6}{7}\pi$$

b) $$\left| \frac{1}{z^2-1} \right| \le \frac{1}{|z|^2-1} = \frac{1}{4-1} = \frac{1}{3}$$

$$\therefore \left| \int_C \frac{dz}{z^2-1} \right| \le \frac{1}{3}\pi$$

2. length of $C = \sqrt{1+1} = \sqrt{2}$

$|z| \ge \left| \frac{1}{2} + \frac{1}{2} i \right| = \frac{1}{\sqrt{2}}$ on C.



$$\therefore \left| \frac{1}{z^4} \right| = \frac{1}{|z|^4} \le \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^4} = 4$$

$$\therefore \left| \int_C \frac{dz}{z^4} \right| \le 4\sqrt{2}$$

4. $$\left| \frac{2z^2-1}{z^4+5z^2+4} \right| = \left| \frac{2z^2-1}{(z^2+4)(z^2+1)} \right| \le \frac{2|z|^2-1}{(|z|^2-4)(|z|^2-1)} = \frac{2R^2-1}{(R^2-4)(R^2-1)}$$ on C_R

length of $C_R = \pi R$

$$\therefore \left| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} dz \right| \leq \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$$
$$= \frac{\frac{1}{R}\pi(2+\frac{1}{R^2})}{(1-\frac{1}{R^2})(1-\frac{4}{R^2})} \to 0 \text{ as } R \to \infty$$

5 $$\left| \frac{\operatorname{Log} z}{z^2} \right| \leq \frac{|\ln r| + |i\theta|}{|z|^2} \leq \frac{\ln R + \pi}{R^2}$$

length of $C_R = 2\pi R$.

$$\therefore \left| \int_{C_R} \frac{\operatorname{Log} z}{z^2} dz \right| \leq 2\pi R \cdot \frac{\ln R + \pi}{R^2} = 2\pi \frac{\ln R + \pi}{R}$$

$$\lim_{R\to\infty} \frac{\ln R + \pi}{R} = \lim_{R\to\infty} \frac{\frac{1}{R}}{1} = 0.$$

6. $$|z^{-\frac{1}{2}} f(z)| = |\exp[-\tfrac{1}{2}\log z]|\,|f(z)|$$
$$= |\exp[-\tfrac{1}{2}\ln|z| - \tfrac{1}{2}i\theta]|\,M, \text{ where } M := \sup_{|z|\leq 1} |f(z)|$$
$$= |\exp[-\tfrac{1}{2}\ln\rho - \tfrac{1}{2}i\theta]|\,M$$
$$= \left|\frac{1}{\sqrt{\rho}} \exp(-\tfrac{1}{2}i\theta)\right| M$$
$$\leq \frac{1}{\sqrt{\rho}} M$$

length of $C_\rho = 2\pi\rho$

$$\therefore \left| \int_{C_\rho} z^{-\frac{1}{2}} f(z)\, dz \right| \leq \frac{1}{\sqrt{\rho}} M \cdot 2\pi\rho = 2\pi\sqrt{\rho} M \to 0 \text{ as } \rho \to 0.$$